Distribution theory June 14, 1999. 9.00-12.00.

Exam paper[1]

I

Consider the differential operator $D = (\frac{d}{dx})^2 - 2\frac{d}{dx} + 1$.

1. Determine the fundamental solution of D belonging to $\mathcal{D}'_+ = \{T \in \mathcal{D}'(\mathbb{R}) : \text{supp}(T) \subset [0, +\infty)\}$.

2. Determine the solution $T \in \mathcal{D}'_+$ of the equation $DT = Y$ (Y the Heaviside one-step function). Hint: use convolution and/or symbolic calculus.

3. Determine, with the help of the Heaviside symbolic calculus, the solution f of the following classical initial value problem, where g is a given continuous function defined on $\mathbb{R}$:

(1) $$Df = g, \; f(0) = 0, \; f'(0) = 1.$$

4. Find the solution f in the case where $g = 1$.

5. Let G be a function of the class C^2 on $\mathbb{R}$. Under which circumstances on G does there exist a continuous function f on $\mathbb{R}$ which satisfies the following convolution equation ? Determine in that case the solution f.

(2) $$\int_0^x f(x-y) y e^y dy = G(x) \qquad x \geq 0$$

II

Notations: $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$. $E = \{T \in \mathcal{D}'(\mathbb{R}) : T_{|\mathbb{R}_*} = \frac{1}{x}\}$.

1. Give the definition of the distribution $\text{pv}\frac{1}{x}$ and show that it belongs to the set E.

2. Determine all distributions belonging to E. Hint: The set $E_0 = \{S \in \mathcal{D}'(\mathbb{R}) : S_{|\mathbb{R}_*} = 0\}$ is known.

3. Show that there exists precisely one distribution T in E which is homogeneous of degree -1 and odd.

III

Let T and T_ε be tempered distributions, $\varepsilon > 0$. One poses by definition $T = \lim_{\varepsilon \to 0} T_\varepsilon$ in $\mathcal{S}'(\mathbb{R})$ if $< T, \varphi > = \lim_{\varepsilon \to 0} < T_\varepsilon, \varphi >$ for all $\varphi \in \mathcal{S}(\mathbb{R})$.

1. Show that this has the following consequence for the Fouriertransform $\mathcal{F}$: $\mathcal{F}(T) = \lim_{\varepsilon \to 0} \mathcal{F}(T_\varepsilon)$.

2. Let T_ε be the regular distribution $Y(x)e^{-\varepsilon x}$. Show that $\lim_{\varepsilon \to 0} T_\varepsilon = Y$ in $\mathcal{S}'(\mathbb{R})$.

3. With the help of this result determine the Fourier transform of Y.

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[1] The parts I, II and III are independent. Clarify your answers by stating the theorems used.